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# A DICHOTOMY FOR PROJECTIONS OF PLANAR SETS

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Abstract. We prove that most one-dimensional projections of a discrete subset of $\mathbb{R}^2$ are either dense in $\mathbb{R}$, or form a discrete subset of $\mathbb{R}$. More precisely, the set E of exceptional directions (for which the indicated dichotomy fails) is a meager subset (of the unit circle $\mathbb{T}$) of Lebesgue measure 0. The set E however does not need to be small in the sense of Hausdorff dimension.

## 1. Main Results.

For $n \geq 1$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, denote by $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{k=1}^{n} x_k y_k$ the standard inner product in $\mathbb{R}^n$, so that $||\boldsymbol{x}|| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$, for $\boldsymbol{x} \in \mathbb{R}^n$.

For $r \geq 0$, denote by $\mathbb{B}^n(r) = \{\boldsymbol{x} \in \mathbb{R}^n \mid ||\boldsymbol{x}|| \leq r\}$ the closed ball of radius $r$ with the center at the origin. A set $M \subset \mathbb{R}^n$ is called discrete if the intersections $M \cap \mathbb{B}^n(r)$ are finite for all $r > 0$.

For $\alpha \in \mathbb{R}$, denote $\boldsymbol{u}_\alpha = (\cos\alpha, \sin\alpha) \in S^1$ where $S^1 = \big\{\boldsymbol{x} \in \mathbb{R}^2 \;\big|\; ||\boldsymbol{x}|| = 1\big\}$ stands for the unit circle. A point in $\boldsymbol{x} \in S^1$ is determined by its direction $\alpha \in \mathbb{T} = [0, 2\pi)$ (so that $\boldsymbol{x} = \boldsymbol{u}_\alpha$).

Denote by $\Phi_\alpha$ the projection map $\Phi_\alpha \colon \mathbb{R}^2 \to \mathbb{R}$ defined by the formula

$$\Phi_\alpha(\boldsymbol{x}) = \boldsymbol{x} \cdot \boldsymbol{u}_\alpha = x_1 \cos\alpha + x_2 \sin\alpha, \qquad \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2. \tag{1.1}$$

**Definition 1.1.** For a set $M \subset \mathbb{R}^2$, define the following three subsets of the set $\mathbb{T} = [0, 2\pi)$ (of directions):

(1) the set of *dense* directions for $M$:
$$\mathrm{DEN}(M) = \big\{\alpha \in \mathbb{T} \;\big|\; \text{the set } \Phi_\alpha(M) \text{ is dense in } \mathbb{R}\big\},$$
(2) the set of *discrete* directions for $M$:
$$\mathrm{DIS}(M) = \big\{\alpha \in \mathbb{T} \;\big|\; \Phi_\alpha(M) \text{ is a discrete subset in } \mathbb{R}\big\},$$
(3) the set of *exceptional* directions for $M$:
$$\mathrm{E}(M) = \mathbb{T} \backslash \big(\mathrm{DEN}(M) \cup \mathrm{DIS}(M)\big).$$

Thus, for every $M \subset \mathbb{R}^2$,

$$\mathbb{T} = \mathrm{DEN}(M) \cup \mathrm{DIS}(M) \cup \mathrm{E}(M) \tag{1.2}$$

is a partition of the set $\mathbb{T}$ of all directions into three distinct subsets.

The central result of the paper is given by the following theorem.

**Theorem 1.1.** *For every discrete subset $M \subset \mathbb{R}^2$, the set of exceptional directions $\mathrm{E}(M)$ is a meager subset of $\mathbb{T}$ of Lebesgue measure 0.*

In other words, the above theorem claims that "most" directions, in both metric and topological senses, are either discrete or dense.

*Remark* 1.1. One easily verifies that, for any (not necessarily discrete) subset $M \subset \mathbb{R}^2$, the sets $\mathrm{DEN}(M)$, $\mathrm{DIS}(M)$ and $\mathrm{E}(M)$ introduced in Definition 1.1 are Borel (see Corollary 2.1 in the next section).

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**Notation.** Through this paper, the following notation is used:

| | | |
|---|---|---|
| **1.** | $\mathrm{DIM_h}(X)$ | stands for the Hausdorff dimension of a set $X \in \mathbb{T}$; |
| **2.** | $\lambda(X)$ | stands for the Lebesgue measure of a set $X \in \mathbb{T}$; |
| **3.** | $\mathbf{card}(X) \le \omega$ | means that $X$ is at most countable. |
| **4.** | $X \triangle Y$ | stands for the symmetric difference $(X \setminus Y) \cup (Y \setminus X)$ of sets $X$ and $Y$; |
| **5.** | $X \equiv Y \pmod{\omega}$ | means that $\mathbf{card}(X \triangle Y) \le \omega$ (i. e., $X$ and $Y$ differ by at most a countable set); |
| **6.** | $X \equiv Y \pmod{\lambda}$ | means that $\lambda(X \triangle Y) = 0$ (i. e., $X$ and $Y$ differ by a set of measure 0). |

*Remark* 1.2. We shall see that there are discrete subsets $M \subset \mathbb{R}^2$ with $\mathrm{DIM_h}(\mathrm{E}(M)) = 1$ (i. e., the exceptional set of direction may have full Hausdorff dimension even though $\lambda(\mathrm{E}(M)) = 0$ by Theorem 1.1). Some examples of such $M$ are given by Propositions 4.1 and 5.1.

Theorem 1.1 admits a generalization for arbitrary (not necessarily discrete) subsets $M \subset \mathbb{R}^2$. This generalization is given by Theorem 1.2 below. We need the following definition.

**Definition 1.2.** For a subset $M \subset \mathbb{R}^2$, define the following three sets:

(1) the set $\mathrm{PB}(M)$ of *P-bounded* directions for $M$:

$$\mathrm{PB}(M) = \Big\{\alpha \in \mathbb{T} \;\Big|\; (\Phi_\alpha)^{-1}(J) \cap M \text{ is bounded in } \mathbb{R}^2, \text{ for every bounded subset } J \subset \mathbb{R}\Big\},$$

(2) the set PUB of *P-unbounded* directions for $M$:

$$\mathrm{PUB}(M) = \Big\{\alpha \in \mathbb{T} \;\Big|\; (\Phi_\alpha)^{-1}(J) \cap M \text{ is unbounded in } \mathbb{R}^2, \text{ for every open non-empty subset } J \subset \mathbb{R}\Big\},$$

(3) the set PE of *P-exceptional* directions for $M$:

$$\mathrm{PE}(M) = \mathbb{T} \setminus \big(\mathrm{PB}(M) \cup \mathrm{PUB}(M)\big). \tag{1.3}$$

**Theorem 1.2.** *For every subset $M \subset \mathbb{R}^2$, the set $\mathrm{PE}(M)$ (of P-exceptional directions) is a meager set of Lebesgue measure 0.*

In other words, "most" directions, in both metric and topological senses, are either $P$-bounded, or $P$-unbounded.

We observe that Theorem 1.2 indeed implies Theorem 1.1 in view of the following relations (taking place for discrete subsets $M \subset \mathbb{R}^2$):

$$\mathrm{E}(M) \subset \mathrm{PE}(M), \qquad \mathbf{card}\big(\mathrm{PE}(M) \setminus \mathrm{E}(M)\big) \le \omega. \tag{1.4}$$

These relations (for discrete $M$) are derived easily from the following ones:

$$\mathrm{PUB}(M) = \mathrm{DEN}(M); \tag{1.5a}$$

$$\mathrm{PB}(M) \subset \mathrm{DIS}(M); \tag{1.5b}$$

$$\mathbf{card}\big(\mathrm{DIS}(M) \setminus \mathrm{PB}(M)\big) \le \omega. \tag{1.5c}$$

While (1.5a) and (1.5b) are obvious, (1.5c) follows from the inclusion $\mathrm{DIS}(M) \setminus \mathrm{PB}(M) \subset W(M)$, where $W(M)$ is the set of all directions determined by pairs of distinct points in $M$, and the fact that $\mathbf{card}(W(M)) \le \omega$ because $\mathbf{card}(M) \le \omega$.

Note that the partition (1.2) is stable (modulo subsets of Lebesgue measure 0) under bounded perturbations of a set $M \subset \mathbb{R}^2$ (Theorem 6.2).

Also, there are multidimensional analogues of Theorems 1.1 and 1.2 (see Section 8).

## 2. Exceptional sets are Borel. Maps $\Psi_\beta$.

It is often more convenient to work with the maps $\Psi_\beta\colon \mathbb{R}^2 \to \mathbb{R}$ defined by the formula

$$\Psi_\beta(\boldsymbol{x}) = x_1 + \beta x_2, \qquad \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2, \qquad (\text{for } \beta \in \mathbb{R}), \tag{2.1}$$

rather than with the maps $\Phi_\alpha$ (see (1.1)).

By analogy with Definition 1.1, for every $M \subset \mathbb{R}^2$, one introduces the sets

$$\mathrm{DEN}'(M) = \{\beta \in \mathbb{R} \mid \Psi_\beta(M) \text{ is dense}\}, \tag{2.2a}$$

$$\mathrm{DIS}'(M) = \{\beta \in \mathbb{R} \mid \Psi_\beta(M) \text{ is discrete}\} \tag{2.2b}$$

and

$$\mathrm{E}'(M) = \mathbb{R}\setminus\big(\mathrm{DIS}'(M) \cup \mathrm{DEN}'(M)\big). \tag{2.2c}$$

The obvious connection

$$\Phi_\alpha(\boldsymbol{x}) = \cos\alpha \cdot \Psi_\beta(\boldsymbol{x}), \qquad \beta = \tan\alpha;$$

implies the equalities

$$\mathrm{DEN}'(M) = \tan(\mathrm{DEN}(M)); \tag{2.3a}$$

$$\mathrm{DIS}'(M) = \tan(\mathrm{DIS}(M)); \tag{2.3b}$$

and

$$\mathrm{E}'(M) = \tan(\mathrm{E}(M)). \tag{2.3c}$$

In view of (2.3a)–(2.3c), the facts that the sets $\mathrm{DEN}(M)$, $\mathrm{DIS}(M)$ and $\mathrm{E}(M)$ are Borel (for an arbitrary subset $M \subset \mathbb{R}^2$) follow immediately from the following theorem.

**Theorem 2.1.** *For every set $M \subset \mathbb{R}^2$, the sets $\mathrm{DEN}'(M), \mathrm{DIS}'(M), \mathrm{E}'(M)$ are Borel subsets of $\mathbb{R}$.*

**Corollary 2.1.** *For every set $M \subset \mathbb{R}^2$, the sets $\mathrm{DEN}(M)$, $\mathrm{DIS}(M)$, $\mathrm{E}(M)$ are Borel subsets of $\mathbb{R}$.*

*Proof of Corollary 2.1.* Follows from Theorem 2.1 and (2.3a)–(2.3c).

□

*Proof of Theorem 2.1.* Denote by $\Sigma$ the family of all rational subintervals of $\mathbb{R}$ (i. e., non-empty subintervals of $\mathbb{R}$ with the rational endpoints). The presentation

$$\mathrm{DEN}'(M) = \bigcap_{J\in\Sigma} \big\{\beta \in \mathbb{R} \mid \Psi_\beta(M) \cap J \neq \emptyset\big\}$$

shows that $\mathrm{DEN}'(M)$ is Borel and in fact a $G_\delta$-set (a countable intersection of open sets).

We assume without loss of generality that $\mathbf{card}(M) \le \omega$. (Otherwise $M$ is replaced by any of its at most countable dense subset; this replacement does not affect the sets $\mathrm{DEN}'(M)$, $\mathrm{DIS}'(M)$ and $\mathrm{E}'(M)$).

Denote by $W(M)$ the set of $\beta \in \mathbb{R}$ for which the map $\Psi_\beta\big|_M\colon M \to \mathbb{R}$ fails to be injective. Then $\mathbf{card}\big(W(M)\big) \le \omega$ because $\mathbf{card}(M) \le \omega$, and the equation $\Psi_\beta(p) = \Psi_\beta(q)$ has at most one solution $\beta \in \mathbb{R}$, for any pair of distinct points $p, q \in M$.

Arrange the countable set $M$ into a sequence $M = \{m_k \mid k \ge 1\}$. One verifies that

$$\mathrm{DIS}'(M)\setminus W(M) = \{\beta \in \mathbb{R} \mid \Psi_\beta(M) \cap J \text{ is finite, for all } J \in \Sigma\} = \bigcap_{\substack{N\ge 1\\ J\in\Sigma}} \Big( \bigcup_{k\ge N} U(k,J)\Big)$$

where all the sets $U(k,J) = \big\{\beta \in \mathbb{R} \mid \Psi_\beta(m_k) \in J\big\}$ are open. Thus $\mathrm{DIS}'(M)\setminus W(M)$ is Borel. Since $\mathbf{card}(W(M)) \le \omega$, $\mathrm{DIS}'(M)$ is also Borel. Finally, $\mathrm{E}'(M)$ is Borel in view of (2.2c).

□

## 3. Proof of Theorem 1.1

In view of (2.3a)–(2.3c), it is enough to prove the following theorem.

**Theorem 3.1.** *For every discrete subset $M \subset \mathbb{R}^2$, the set $\mathrm{E}'(M)$ (defined by (2.3c)) is a meager subset of $\mathbb{T}$ of Lebesgue measure 0.*

*Proof.* Denote by $\Sigma$ the family of all rational subintervals of $\mathbb{R}$ (i. e., non-empty subintervals of $\mathbb{R}$ with the rational endpoints). For any two finite subintervals $P, Q \in \Sigma$, define the set

$$V(P,Q) = \Big\{\beta \in \mathbb{R} \;\Big|\; \Psi_\beta(M) \cap P \text{ is infinite, and } \Psi_\beta(M) \cap Q = \emptyset\Big\}. \tag{3.1}$$

For $\beta \in \mathbb{R}$, the condition $\beta \in \mathrm{E}'(M)$ is equivalent to the existence of two finite interval $P$ and $Q$ (without loss of generality, $P, Q \in \Sigma$) such that $\beta \in V(P,Q)$. (Indeed, the existence of $P$ means that $\beta \notin \mathrm{DIS}'(M)$, and the existence of $Q$ is equivalent to the condition $\beta \notin \mathrm{DEN}'(M)$).

Thus $\mathrm{E}'(M)$ can be represented as the countable union

$$\mathrm{E}'(M) = \bigcup_{P,Q\in\Sigma} V(P,Q).$$

To complete the proof of Theorem 3.1, it remain to verify that every set $V(P,Q)$ is nowhere dense and has Lebesgue measure 0.

Fix $\beta \in V(P,Q)$, $\beta \neq 0$. Since $\Psi_\beta(M) \cap P$ is infinite, there exists an infinite sequence of distinct points $\boldsymbol{z}_k = (x_k, y_k) \in M \subset \mathbb{R}^2$, $k \geq 1$, such that $\Psi_\beta(\boldsymbol{z}_k) = x_k + \beta y_k \in P$. Since $M$ is discrete,

$$\lim_{k\to\infty} ||\boldsymbol{z}_k|| \to \infty.$$

Moreover, we have

$$\lim_{k\to\infty} |x_k| = \lim_{k\to\infty} |y_k| = \infty$$

because the points $\Psi_\beta(\boldsymbol{z}_k) = x_k + \beta y_k \in P$ lie in the (bounded) interval $P$, and $\beta \neq 0$.

We may assume that all $y_k \neq 0$ (by dropping a few first terms of $\{\boldsymbol{z}_k\}$ if needed). Denote $\varepsilon_k = |y_k|^{-1}$. Consider two sequences of intervals:

$$P_k = \Big\{t \in \mathbb{R} \;\Big|\; \Psi_t(\boldsymbol{z}_k) = x_k + t y_k \subset P\Big\} = (P - x_k)\,\varepsilon_k$$
$$Q_k = \Big\{t \in \mathbb{R} \;\Big|\; \Psi_t(\boldsymbol{z}_k) = x_k + t y_k \subset Q\Big\} = (Q - x_k)\,\varepsilon_k.$$

Set

$$d = \mathbf{diam}(P \cup Q); \qquad q \;= \mathbf{diam}(Q) \;= \lambda(Q);$$
$$d_k = \mathbf{diam}(P_k \cup Q_k); \qquad q_k = \mathbf{diam}(Q_k) = \lambda(Q_k);$$

where $\mathbf{diam}(A)$ stands for the diameter of a set $A$. Clearly $q_k = q \cdot \varepsilon_k$ and $d_k = d \cdot \varepsilon_k$.

Since $\beta \in V(P,Q)$, for all $k \geq 1$, the relations

$$\beta \in P_k \quad \text{and} \quad Q_k \cap V(P,Q) = \emptyset$$

hold by the definition of $V(P,Q)$ (see (3.1)).

We observe that, for every $k \geq 1$, the $d_k$-neighborhood

$$J_k = \big(\beta - d_k, \beta + d_k\big)$$

of $\beta$ contains a subinterval $Q_k$ of length $q_k$ which does not intersect $V(P,Q)$. Note that $\lambda(J_k) = 2d_k \to 0$ as $k \to 0$, and the ratio $\frac{\lambda(Q_k)}{\lambda(J_k)} = \frac{q}{2d} \leq \frac{1}{2}$ does not depend on $k$.

It has been shown that for every $\beta \in V(P,Q)\setminus\{0\}$ one can find arbitrary small intervals $J_k$ around $\beta$ which contains a further subinterval $Q_k$ of relative density $\frac{q}{2d}$ such that $Q_k \cap V(P,Q) = \emptyset$.

The above property implies that the set $V(P,Q)$ is nowhere dense and has Lebesgue measure 0 (because it has no Lebesgue density points). The proof of Theorem 3.1 is complete. □

## 4. Exceptional set E($M$) may have full Hausdorff dimension

The following proposition provides an example of a discrete set $M \subset \mathbb{R}^2$ whose exceptional set $\mathrm{E}(M)$ has full Hausdorff dimension. A class of examples of such $M$ will be presented in the next section.

**Proposition 4.1.** *For the set*

$$M_0 = \{(m^2, n^2) \mid m, n \in \mathbb{N} = 1, 2, 3, \ldots\} \subset \mathbb{R}^2,$$

*the exceptional set has full Hausdorff dimension:* $\dim_{\mathrm{H}}(\mathrm{E}(M_0)) = 1$.

In view of (2.3c), it is enough to show that $\dim_{\mathrm{H}}(\mathrm{E}'(M_0)) = 1$.

For $\alpha \in \mathbb{R}$, denote

$$L(\alpha) = \Psi_\alpha(M_0) \subset \mathbb{R} = \big\{\alpha m^2 - n^2 \mid m, n \in \mathbb{N}\big\}. \tag{4.1}$$

Then (see notation in (2.2a)–(2.2c))

$$\mathrm{DEN}'(M_0) = \{\alpha \in \mathbb{R} \mid L(\alpha) \text{ is dense in } \mathbb{R}\}, \tag{4.2a}$$
$$\mathrm{DIS}'(M_0) = \{\alpha \in \mathbb{R} \mid L(\alpha) \text{ is discrete in } \mathbb{R}\},$$

and

$$\mathrm{E}'(M_0) = \mathbb{R} \setminus \big(\mathrm{DIS}'(M_0) \cup \mathrm{DEN}'(M_0)\big). \tag{4.2b}$$

**Lemma 4.1.** *No irrational* $\alpha > 0$ *lies in* $\mathrm{DIS}'(M_0)$.

*Proof.* Assume to the contrary that some irrational $\alpha \in \mathrm{DIS}'(M_0)$. Then the number $\beta = \sqrt{\alpha}$ is also irrational, and the diophantine inequality $|m\beta - n| < \frac{1}{m}$ has infinitely many solutions in $m, n \in \mathbb{Z}^2, m \geq 1$. For each such a solution, we have

$$\big|\alpha m^2 - n^2\big| = \big|m\beta - n\big| \cdot \big|m\beta + n\big| < \frac{|m\beta + n|}{m} =$$
$$= \frac{\big|2m\beta - (m\beta - n)\big|}{m} < 2\beta + 1.$$

Sinve $\alpha$ is irratinal, we conclude that $L(\alpha)$ contains infinitely many points in the finite interval $(-(2\beta+1), 2\beta+1)$. Thus $L(\alpha)$ is not discrete, a contradiction with the assumption that $\alpha \in \mathrm{DIS}'(M_0)$. □

For $x \in \mathbb{R}$, denote by $\langle x \rangle$ the distance from $x$ to the closest integer: $\langle x \rangle = \mathbf{dist}(x, \mathbb{Z}) = \min\limits_{k \in \mathbb{Z}} |x - k|$.

Let $\mathbb{N} = \{k \in \mathbb{Z} \mid k \geq 1\} = \{1, 2, \ldots\}$ stand for the set of natural numbers. A number $\alpha \in \mathbb{R}$ is called *badly approximable* if there exists an $\varepsilon > 0$ such that $m\langle m\alpha \rangle > \varepsilon$, for all $m \in \mathbb{N}$. Denote by BA the set of badly approximable numbers. It is clear that this set does not contain rationals: $\mathrm{BA} \cap \mathbb{Q} = \emptyset$.

**Lemma 4.2.** *If* $\beta \in \mathrm{BA}$ *then* $\alpha = \beta^2 \notin \mathrm{DEN}'(M_0)$.

*Proof.* Since $\beta \in \mathrm{BA}$, there exists an $\varepsilon > 0$ such that $m\langle m\alpha \rangle > \varepsilon$, for all $m \in \mathbb{N}$. Without loss of generality, one assumes that $\beta > 0$. Then, for any $m, n \in \mathbb{N}$, the following inequality holds:

$$\big|\alpha m^2 - n^2\big| = \big|m\beta - n\big| \cdot \big|m\beta + n\big| \geq \langle m\beta \rangle \cdot \big|m\beta + n\big| > \frac{\varepsilon\,|m\beta + n|}{m} > \varepsilon\beta.$$

We observe that $L(\alpha) \cap \big(-\varepsilon\beta, \varepsilon\beta\big) = \emptyset$ (see (4.1)), whence $\alpha \notin \mathrm{DEN}'(M_0)$ (by (4.2a)), a contradiction. □

**Corollary 4.1.** *Assume that* $\alpha > 0$ *is irrational such that* $\beta = \sqrt{\alpha} \in \mathrm{BA}$ *(i. e., that* $\beta$ *is badly approximable). Then* $\alpha \in \mathrm{E}'(M_0)$.

*Proof.* One derives $\alpha \notin \mathrm{DIS}'(M_0)$ and $\alpha \notin \mathrm{DEN}'(M)$ from Lemmas 4.1 and 4.2, respectively. Therefore $\alpha \in \mathrm{E}'(M_0)$, in view of (4.2b). □

Denote by $C = (\mathrm{BA})^2 \backslash \mathbb{Q}$ the set of irrational squares of badly approximable numbers. By Corollary 4.1, $C \subset \mathrm{E}'(M_0)$. It is well known that $\dim_{\mathrm{H}}(BA) = 1$. (The set $\mathrm{BA} \subset \mathbb{R}$ of badly approximable numbers has full Hausdorff dimension, see e. g. [11]).

It follows that $1 \geq \dim_{\mathrm{H}}(\mathrm{E}'(M_0)) \geq \dim_{\mathrm{H}}(C) = 1$ and hence $\dim_{\mathrm{H}}(\mathrm{E}(M_0)) = 1$ (in view of (2.3c)). This completes the proof of Proposition 4.1.

*Remark* 4.1. The above arguments coupled with Theorem 1.1 provide a short proof of the known fact that the set BA of badly approximable numbers has Lebesgue measure 0.

## 5. More examples

A sequence $\boldsymbol{r} = \{r_k\}_{k\geq 1}$ of real numbers is said to be *rising to infinity* if it is strictly increasing and if $\lim\limits_{k\to\infty} r_k = \infty$. For any such a sequence $\boldsymbol{r}$, define the discrete set

$$M(\boldsymbol{r}) = \big\{(n, r_k) \mid n, k \in \mathbb{Z}, k \geq 1\big\} \subset \mathbb{R}^2. \tag{5.1}$$

By Theorem 1.1, $\lambda(\mathrm{E}(M(\boldsymbol{r}))) = 0$.

**Proposition 5.1.** *Let $\boldsymbol{r} = \{r_k\}_{k\geq 1}$ be a rising to infinity sequence of real numbers. Then*

1. *If $\boldsymbol{r}$ is lacunary (i. e., if $\liminf\limits_{k\geq 1} \frac{r_{k+1}}{r_k} > 1$) then $\mathrm{DIM}_{\mathrm{h}}(\mathrm{E}(M(\boldsymbol{r}))) = 1$.*
2. *If $\boldsymbol{r}$ is sublacunary (i. e., if $\lim\limits_{k\to\infty} \frac{r_{k+1}}{r_k} = 1$) then $\mathrm{DIM}_{\mathrm{h}}(\mathrm{E}(M(\boldsymbol{r}))) = 0$.*
3. *If $\limsup\limits_{k\to\infty} (r_{k+1} - r_k) < \infty$ then $\mathbf{card}(\mathrm{E}(M(\boldsymbol{r}))) \leq \omega$ (i. e., the set $\mathrm{E}(M(\boldsymbol{r}))$ is at most countable).*

In what follows in this section, we assume that $\boldsymbol{r} = \{r_k\}_{k\geq 1}$ be a rising to infinity sequence. One easily verifies that $\mathrm{PB}(M(\boldsymbol{r})) = \emptyset$, and hence (see (1.5c))

$$\mathbf{card}(\mathrm{DIS}(M(\boldsymbol{r}))) \leq \omega. \tag{5.2}$$

The following proposition follows immediately from Theorem 1.1. Recall that a set is called residual if its complement is meager.

**Proposition 5.2.** $\mathrm{DEN}(M(\boldsymbol{r})) \subset \mathbb{T} = [0, 2\pi)$ *is a residual set of full Lebesgue measure (in $\mathbb{T}$).*

Taking in account (2.3a), we conclude the following.

**Corollary 5.1.** $\mathrm{DEN}'(M(\boldsymbol{r})) \subset \mathbb{R}$ *is a residual set of full Lebesgue measure (in $\mathbb{R}$).*

On the other hand, the set $\mathrm{DEN}'(M(\boldsymbol{r}))$ may be defined in the following way:

$$\begin{aligned} \mathrm{DEN}'(M(\boldsymbol{r})) &= \Big\{\beta \in \mathbb{R} \mid \text{ the set } \big\{(n + \beta r_k) \mid n, k \in \mathbb{Z}, k \geq 1\big\} \text{ is dense in } \mathbb{R}\Big\} = \\ &= \Big\{\beta \in \mathbb{R} \mid \text{ the sequence } \big\{\beta r_k\big\} \text{ is dense } \pmod 1\Big\}, \end{aligned} \tag{5.3}$$

Now we are ready to derive the following (known) result.

**Proposition 5.3.** *If a set $S \subset \mathbb{R}$ is unbounded, then the set*

$$\mathrm{ND}(S) = \big\{\beta \in \mathbb{R} \;\big|\; \beta \cdot S \text{ is not dense } (\mathrm{mod}\ 1)\big\}$$

*is a meager subset of $\mathbb{R}$ of Lebesgue measure* 0.

Proposition 5.3 follows easily from classical uniform distribution results (see e. g. [5, Ch.1, §4, Cor. 4.3]). For a direct simple proof see [1, §6]. What follows is a derivation of Proposition 5.3 from Theorem 1.1.

**Proof of Proposition 5.3.** Let assume for definiteness that the set $S$ is unbounded from above. Then there exists a rising to infinity sequence $\boldsymbol{r} = \{r_k\}_{k\geq 1}$ of positive reals in $S$.

Let $M(\boldsymbol{r})$ be defined as in (5.1). Since the set $\mathrm{ND}(S)$ is a subset of the complement of the set $\mathrm{DEN}'(M(\boldsymbol{r}))$ in (5.3), the claim of the proposition follows from Corollary 5.1. □

**Proof of Proposition 5.1.** We demonstrate that Proposition 5.1 is just a reformulation of some known results on distribution mod 1 of certain sequences of reals. Denote

$$\mathrm{ND}(\boldsymbol{r}) = \Big\{\beta \in \mathbb{R} \;\Big|\; \text{the sequence } \big\{\beta \cdot r_k\big\} \text{ is not dense } (\mathrm{mod}\ 1)\Big\}.$$

Since $\mathrm{ND}(\boldsymbol{r})$ is the complement of the set $\mathrm{DEN}'(M(\boldsymbol{r}))$ in $\mathbb{R}$, we obtain

$$\mathrm{E}'(M(\boldsymbol{r})) \subset \mathrm{ND}(\boldsymbol{r}), \qquad \mathrm{ND}(\boldsymbol{r}) \backslash \mathrm{E}'(M(\boldsymbol{r})) = \mathrm{DIS}'(M(\boldsymbol{r})).$$

where $\mathbf{card}(\mathrm{DIS}'(M(\boldsymbol{r}))) \le \omega$, in view of (5.2) and (2.3b).

We conclude that

$$\mathrm{DIM_h}(\mathrm{ND}(\boldsymbol{r})) = \mathrm{DIM_h}(\mathrm{E}'(M(\boldsymbol{r}))) = \mathrm{DIM_h}(\mathrm{E}(M(\boldsymbol{r}))), \tag{5.4a}$$

and that (see (2.3c))

$$\mathbf{card}(\mathrm{ND}(\boldsymbol{r})) \le \omega \implies \mathbf{card}(\mathrm{E}'(M(\boldsymbol{r}))) \le \omega \implies \mathbf{card}(\mathrm{E}(M(\boldsymbol{r}))) \le \omega. \tag{5.4b}$$

It is known ([6], [7], [9]) that if $\boldsymbol{r} = \{r_k\}_{k\ge 1}$ is a lacunary sequence of positive integers, then the set $\mathrm{ND}(\boldsymbol{r})$ has full Hausdorff dimension: $\mathrm{DIM_h}(\mathrm{ND}(\boldsymbol{r})) = 1$. The claim (1) of Proposition 5.1 now follows from (5.4a).

On the other hand, by [1, Theorem 1.3] $\mathrm{DIM_h}(\mathrm{ND}(\boldsymbol{r})) = 0$ for sublacinary sequences $\boldsymbol{r} = \{r_k\}_{k\ge 1}$ rising to infinity. The claim (2) of Proposition 5.1 also follows from (5.4a).

Finally, the claim (3) of Proposition 5.1 follows from (5.4b) and the fact that, under the current assumption that $\limsup_{k\to\infty} (r_{k+1} - r_k) < \infty$, the set $\mathrm{ND}(\boldsymbol{r})$ must be at most countable [1, Proposition 1.8]. (In the special case when all $r_k$ are integers the fact is also proved in [10, Corollaries 42 and 43]). □

## 6. Projections of arbitrary sets. Proof of Theorem 1.2

The relations (1.5a)–(1.5c) mean, in particular, that, for discrete $M \subset \mathbb{R}^2$, the partition (1.2) coincides (modulo countable sets) with the partition

$$\mathbb{T} = \mathrm{PUB}(M) \cup \mathrm{PB}(M) \cup \mathrm{PE}(M) \tag{6.1}$$

The proof of Theorem 1.2 (that the set $\mathrm{PE}(M)$ must be small in both metric and topological senses) is based on Theorems 1.1 and Theorem 6.1 which asserts stability of $\mathrm{PB}(M)$ under bounded perturbations of subsets $M \subset \mathbb{R}^2$.

For $r > 0$ and a set $A \subset \mathbb{R}^k$, we use the standard notation

$$\mathcal{N}(r, A) \stackrel{\mathrm{def}}{=} \{\boldsymbol{x} \in \mathbb{R}^k \mid \mathbf{dist}(A, \boldsymbol{x}) < r\}, \tag{6.2}$$

for the $r$-neighborhood of $A$ where $\mathbf{dist}(A, \boldsymbol{x}) \stackrel{\mathrm{def}}{=} \inf\limits_{a\in A} ||x - a|| \in [0, \infty]$.

For two subsets $A, B \subset \mathbb{R}^2$ define

$$D_0(A, B) = \sup_{b\in B} \mathbf{dist}(A, b) = \inf\Big(\big\{r > 0 \mid B \subset \mathcal{N}(r, A)\big\}\Big) \in \big[0, \infty\big]. \tag{6.3}$$

Recall that the *Hausdorff distance* $\mathrm{D_H}(M_1, M_2)$ between two non-empty sets $M_1, M_2 \in \mathbb{R}^2$ is defined as

$$\mathrm{D_H}(M_1, M_2) = \max\big(D_0(M_1, M_2),\, D_0(M_2, M_1)\big) \in \big[0, \infty\big]. \tag{6.4}$$

**Theorem 6.1.** *Assume that* $\mathrm{D_H}(M_1, M_2) < \infty$ *for two subsets* $M_1, M_2 \subset \mathbb{R}^2$. *Then*

$$\mathrm{PB}(M_1) = \mathrm{PB}(M_2).$$

Theorem 6.1 follows immediately from the following

**Proposition 6.1.** *Assume that for some subsets* $M_1, M_2 \subset \mathbb{R}^2$

$$D_0(M_1, M_2) = \sup_{x\in M_2} \mathbf{dist}(x, M_1) < \infty.$$

*Then* $\mathrm{PB}(M_1) \subset \mathrm{PB}(M_2)$.

*Proof of Proposition 6.1.* Denote $u = D_0(M_1, M_2)$ and select any $v > u \geq 0$. Then $M_2 \subset \mathcal{N}(v, M_1)$.

Given $\alpha \in \mathrm{PB}(M_1)$, we have to show that $\alpha \in \mathrm{PB}(M_2)$. This is to say that, for every bounded interval $J = (a, b) \subset \mathbb{R}$, the set $K = (\Phi_\alpha)^{-1}(J) \cap M_2$ is bounded in $\mathbb{R}$.

Denote by $J'$ the open interval $J' = \mathcal{N}(v, J) = (a - v, b + v)$. Then we have (see notation (6.2))

$$K \;=\; (\Phi_\alpha)^{-1}(J) \cap M_2 \subset\; (\Phi_\alpha)^{-1}(J) \cap \mathcal{N}(v, M_1) \subset$$
$$\subset\; \mathcal{N}(v, (\Phi_\alpha)^{-1}(J') \cap M_1) \stackrel{\mathrm{def}}{=} L.$$

Since $\alpha \in \mathrm{PB}(M_1)$, the set $P = (\Phi_\alpha)^{-1}(J') \cap M_1$ is bounded. It follows that the sets $L = \mathcal{N}(v, P)$ and $K \subset L$ are also bounded, completing the proof of Proposition 6.1. □

*Proof of Theorem 6.1.* Since $\mathrm{D_H}(M_1, M_2) < \infty$, both sets $D_0(M_1, M_2)$ and $D_0(M_2, M_1)$ are finite. Apply Proposition 6.1. □

*Proof of Theorem 1.2.* There exists a subset $M' \subset M$ such that $M \subset \mathcal{N}(1, M')$ and $M'$ is discrete in $\mathbb{R}^2$. (Take a subset $M' \subset M$ which is maximal under the constraint that the distances between any distinct points of $M'$ are $\leq 1$.

Let $U = \mathrm{DIS}(M')\backslash\mathrm{PB}(M')$. In view of (1.5b) and (1.5c), we have

$$\mathbf{card}(U) \leq \omega \qquad \text{and} \qquad \mathrm{PB}(M') = \mathrm{DIS}(M')\backslash U.$$

On the other hand, it follows from (1.5a) and the inclusion $M' \subset M$ that

$$\mathrm{DEN}(M') = \mathrm{PUB}(M') \subset \mathrm{PUB}(M).$$

We obtain (see (1.3)):

$$\mathrm{PE}(M) = \mathbb{T}\backslash\big(\mathrm{PB}(M) \cup \mathrm{PUB}(M)\big) \subset \mathbb{T}\backslash\Big(\big(\mathrm{DIS}(M)\backslash U\big) \cup \mathrm{DEN}(M')\Big) \subset \mathrm{E}(M') \cup U.$$

The claim of Theorem 1.2 follows from Theorem 1.1 and the fact that $\mathbf{card}(U) \leq \omega$. □

The following theorem summarizes the results on stability of the partitions (1.2) and (6.1) (see notation following Remark 1.2), under bounded perturbations of a set $M$. Recall that $\mathrm{D_H}(\,\cdot\,,\,\cdot\,)$ stands for the Haudorff distance between sets, see (6.4).

**Theorem 6.2** (Summary)**.** *Assume that* $\mathrm{D_H}(M_1, M_2) < \infty$*, for subsets* $M_1, M_2 \subset \mathbb{R}^2$*. Then:*

**(1)** $\mathrm{PB}(M_1) = \mathrm{PB}(M_2)$*;*

**(2)** $\lambda(\mathrm{PE}(M_i)) = 0$*, for both* $i = 1, 2$*;*

**(3)** $\mathrm{PUB}(M_1) \equiv \mathrm{PUB}(M_2) \pmod{\lambda}$*;*

*If, moreover, both* $M_1, M_2$ *are discrete then*

**(4)** $\mathrm{DIS}(M_1) \equiv \mathrm{DIS}(M_2) \equiv \mathrm{PB}(M_2) \pmod{\omega}$*;*

**(5)** $\mathrm{DEN}(M_1) \equiv \mathrm{DEN}(M_2) \equiv \mathrm{PUB}(M_2) \pmod{\lambda}$.

*Proof of Theorem 6.2.* **(1)** and **(2)** are exactly Theorems 6.1 and 1.2, respectively. **(3)** follows from **(1)**. Finally, **(4)** follows from **(1)**, (1.5b) and (1.5c), and **(5)** follows from **(3)** and (1.5a). □

We derive the following corollary for projections of syndetic subset of $\mathbb{R}^2$. A set $M \subset \mathbb{R}^2$ is called *syndetic* if $\mathrm{D_H}(M, \mathbb{R}^2) = D_0(M, \mathbb{R}^2) < \infty$ (see (6.3) and (6.4)).

**Proposition 6.2.** *For a syndetic subset* $M \subset \mathbb{R}^2$*, one has*

$$\mathrm{PUB}(M) \equiv \mathrm{DEN}(M) \equiv \mathbb{T} = [0, 2\pi).$$

*Proof.* By Theorem 6.2, **(3)** , $\mathbb{T} = \mathrm{PUB}(\mathbb{R}^2) \equiv \mathrm{PUB}(M) \pmod{\lambda}$ whence $\mathrm{DEN}(M) = \mathbb{T} \pmod{\lambda}$, in view of (1.5a). (We may assume that $M$ is discrete because otherwise one replaces $M$ with its syndetic discrete subset. □

*Remark* 6.1. One easily verifies that, for syndetic subsets $M \subset \mathbb{R}^2$, one has $\mathrm{PB}(M) = \emptyset$, and that $\mathbf{card}(\mathrm{DIS}(M) \le \omega$. There are examples of discrete sundetic subsets $M \subset \mathbb{R}^2$ for which $\mathrm{D_H}(E(M) = 1$. On the other hand, one can prove that, for every discrete sundetic subsets $M \subset \mathbb{R}^2$, the exceptional set $\mathrm{E}(M)$ is a countable union of sets of the box dimension $< 1$. (In fact, then $\mathrm{E}(M)$ must be a countable union of perforated sets in the sense of [1, §3]). Under the additional assumption on $M$ to be periodic (there exists $\boldsymbol{x} \in \mathbb{R}^2 \setminus \{0\}$ such that $M + \boldsymbol{x} = M$), $\mathbf{card}(\mathrm{E}(M)) \le \omega$.

## 7. Some questions

Denote $\mathbb{R}^+ = [0, \infty)$. Given a continuous function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$, a subset $M \subset \mathbb{R}^2$ is said to be $g$-syndetic if the set

$$\mathbf{F}_g = \big\{\boldsymbol{x} \in \mathbb{R}^2 \mid \mathbf{dist}(\boldsymbol{x}, M) \ge g(|\boldsymbol{x}|)\big\}$$

is bounded. If the relation $\mathrm{DEN}(M) \equiv \mathbb{T} \pmod{\lambda}$ holds for all $g$-syndetic discrete subsets $M \subset \mathbb{R}^2$, then the function $g$ is called P-negligible. By Proposition 6.2, bounded $g$ must be P-negligible.

*Question* 7.1. Does there exist P-negligible function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim\limits_{x\to\infty} g(x) = \infty$?

One can speculate that $g(x) = \sqrt{x}$ is P-negligible. This is in agreement with fact that the set

$$M = \big\{(\pm m^2, \pm n^2) \mid m, n \in \mathbb{N}\big\}$$

satifies $\mathrm{DEN}(M) \equiv \mathbb{T} \pmod{\lambda}$ (cf. Proposition 4.1 and its proof in Section 4). On the other hand, the functions $g(x) = x^a$ with $a > 1/2$ fail to be P-negligible as it follows from the following proposition.

**Proposition 7.1.** *For $a > 0$, denote $M(a) = \big\{(\pm m^a, \pm n^a) \mid m, n \in \mathbb{N}\big\} \subset \mathbb{R}^2$. Then*

$$\mathrm{DIS}(M(a)) \equiv \mathbb{T} \pmod{\lambda}, \qquad \textit{for } u > 2,$$

*and*

$$\mathrm{DEN}(M(a)) \equiv \mathbb{T} \pmod{\lambda}, \qquad \textit{for } u \le 2.$$

The proof of Theorem 7.1 is based on Theorem 1.1 and the fact that, for Lebsgue almost all $t \in \mathbb{R}$, the inequality $m^{a-1}\langle mt\rangle < 1$ has a finite or infinite number of solutions $m \in \mathbb{Z}$ depending on whether or not $a > 2$ (cf. proof of Lemma 4.1 in Section 4).

**Definition 7.1** (Notation)**.** *Denote by $\mathcal{P}$ the family of measurable subsets $A \subset \mathbb{T}$ for which there exists a set $M \subset \mathbb{R}^2$ such that $\mathrm{DEN}(M) \equiv A \pmod{\lambda}$ (see notation following Remark 1.2).*

Note that requiring sets $M$ to be discrete (in the above definition) would not affect the defined family $\mathcal{P}$ because of **(5)** in Theorem 6.2).

The problem of characterization of sets in the family $\mathcal{P}$ is open. Clearly, a set $A \in \mathcal{P}$ must be Lebesgue measurable and $\pi$-periodic (which means $A + \pi \equiv A \pmod{\lambda}$).

*Question* 7.2. Does the family $\mathcal{P}$ coincide with the family of all $\pi$-periodic measurable subsets of $T$?

We observe that any $\pi$-periodic finite union $A$ of subintervals of $\mathbb{T}$ must lie in $\mathcal{P}$. One just takes

$$M_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 \;\middle|\; x_1 \ne 0,\ \frac{x_2}{x_1} \in A\right\}$$

(or $M_2 = M_1 \cap (\mathbb{Z} \times \mathbb{Z})$ to make the set $M$ discrete).

## 8. Multidimensional extensions

Main results of the paper (Theorems 1.1, 1.2 and 6.2) extend to all dimensions $n \ge 2$. The following is a multidimensional version of Theorem 1.1.

**Theorem 8.1.** *Let $n > k \ge 1$ be integers and let $M \subset \mathbb{R}^n$ be a discrete subset. Then, for Lebesgue almost all $k$-planes $U \subset \mathbb{R}^n$ (in the sense of the natural $k(n-k)$-Lebesgue measure on the Grassmannian $\mathbf{GR}(k, n)$) the projection of $M$ on $U$ is either dense in $U$, or discrete in $U$.*

The proof of Theorem 8.1 is more complicate and longer than that of Theorem 1.1, and it is not included (even though the basic idea is the same). It will be published elsewhere in the case an application of Theorem 8.1 justifying the length of the proof will be found.

The statements of the multidimensional versions of Theorems 1.2 and 6.2 are straightforward, and we omit these.

## 9. Projection of random sets in $\mathbb{R}^2$

We conclude the paper by formulating results (also without proofs) on the generic size of the sets $\mathrm{DIS}(M)$, $\mathrm{DEN}(M)$ and $\mathrm{E}(M)$, for random subset $M \subset \mathbb{R}^2$ of given density. We consider two settings.

9.1. **First setting.** Denote by $\boldsymbol{\alpha}$ the sequence $\{\alpha_k\}$ of independent random variables, each uniformly distributed in $\mathbb{T} = [0, 2\pi)$. For every increasing to infinity sequence $\boldsymbol{r} = \{r_k\}_{k\geq 1}$ of positive numbers, consider the random set $M = \{\boldsymbol{x}_k \mid k \geq 1\}$ of points in $\mathbb{R}^2$ where

$$\boldsymbol{x}_k = (r_k \cos \alpha_k, r_k \sin \alpha_k) \in \mathbb{R}^2$$

is the point with polar coordinates $(r_k, \alpha_k)$. Thus the points $\boldsymbol{x}_k$, $k \geq 1$, are selected independently, each point $\boldsymbol{x}_k$ being picked up randomly on the circle $\big\{\boldsymbol{x} \in \mathbb{R}^2 \;\big|\; ||x|| = r_k\big\}$.

The following theorem describes the generic size of the sets $\mathrm{DIS}(M)$, $\mathrm{DEN}(M)$ and $\mathrm{E}(M)$ for the discrete random set $M = \{\boldsymbol{x}_k \mid k \geq 1\} \in \mathbb{R}^2$. A statement is said to be satisfied a. s. (almost sure) if it holds for almost all choices of the sequence $\boldsymbol{\alpha}$.

**Theorem 9.1.** *Given an increasing to infinity sequence $\boldsymbol{r} = \{r_k\}_{k\geq 1}$ of positive numbers, the following statements take place:*

(1) $\mathrm{DEN}(M)$ *is a residual subset of* $\mathbb{T}$, *a. s.*

(2) *The relation* $\mathrm{DEN}(M) = \mathbb{T} \pmod{\lambda}$ *a. s. takes place if and only if* $\sum_{k\geq 1} \frac{1}{r_k} = \infty$.

(3) *The relation* $\mathrm{DIS}(M) = \mathbb{T} \pmod{\lambda}$ *a. s. takes place if and only if* $\sum_{k\geq 1} \frac{1}{r_k} < \infty$.

(4) *The relation* $\mathrm{DIS}(M) = \mathbb{T}$ *a. s. takes place if and only if* $\limsup\limits_{N\to\infty} \Big( \sum\limits_{k=1}^{N} \dfrac{1}{r_k \cdot \log N} \Big) = \infty$.

*Proof.* Statesments (2) and (3) are obtained by standard application of Borel-Cantelli lemma. One also verifies that $\mathrm{DEN}(M)$ is a residual subset of $\mathbb{T} = [0, 2\pi)$ whenever $\boldsymbol{\alpha}$ is dense in $\mathbb{T}$, an a. s. condition. This proves (1).

Statement (4) is more delicate. Some readers may find it surprising that the conditions on sequence $\boldsymbol{r} = \{r_k\}$ for (2) and (4) are not equivalent. The situation here is reminiscent to Dvoretzky's problem on covering the circle by random arcs (see [4, Ch. 11], [12] for the description of the problem and its solution by L. Shepp). Statement (4) can be derived from the solution of Dvoretzky's problem.

□

## 10. Concluding remarks

The main result of the paper (Theorem 1.1) was inspired by a conversation with Hillel Furstenberg in 1993. A shorter version of the present work was circulating as an unpublished preprint of 1994.

I would like to thank Benjy Weiss and Hillel Furstenberg for useful discussions (conducted years ago) and also Yuval Peres for his encouragement to publish the results of this paper. (His 2000 paper [8], joint with Boris Solomyak, refers to the unpublished preprint mentioned above).

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